Exercise 59

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x^{-2} \ln x, \quad \left[\frac{1}{2}, 4\right]$$

Solution

Take the derivative of the function.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x^{-2} \ln x) \\
&= \left[\frac{d}{dx}(x^{-2})\right] \ln x + x^{-2} \left[\frac{d}{dx}(\ln x)\right] \\
&= (-2x^{-3}) \ln x + x^{-2} \left(\frac{1}{x}\right) \\
&= -\frac{2 \ln x}{x^3} + \frac{1}{x^3} \\
&= \frac{1 - 2 \ln x}{x^3}
\end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve both equations for x.

$$\begin{aligned}
1 - 2 \ln x &= 0 & x^3 &= 0 \\
\ln x &= \frac{1}{2} & x &= 0 \\
x = e^{1/2} &\approx 1.64872 & x &= 0
\end{aligned}$$

Only $x = e^{1/2}$ is within $[0.5, 4]$, so evaluate f here.

$$f(e^{1/2}) = (e^{1/2})^{-2} \ln e^{1/2} = \frac{1}{2e} \approx 0.18394 \qquad \text{(absolute maximum)}$$

Now evaluate the function at the endpoints of the interval.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-2} \ln \frac{1}{2} = -4 \ln 2 \approx -2.77259 \qquad \text{(absolute minimum)}$$

$$f(4) = (4)^{-2} \ln 4 \approx 0.0866434$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $\left[\frac{1}{2}, 4\right]$.

The graph of the function below illustrates these results.